# AN EFFICIENT ALGORITHM FOR ZEROS OF BOUNDED GENERALIZED GHI-QUASI-ACCRETIVE MAPS

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## ABSTRACT.

This is a research announcement of the following result. Let $E$ be a real normed linear space in which the single-valued normalized duality map is Holder continuous on balls and let $A\colon E \to E$ be a bounded generalized $\Phi$-quasi-accretive map. A Mann-type iterative sequence is constructed and proved to converge strongly to the unique zero of $A$. In particular, our Theorems are applicable in real Banach spaces that include the $L_p$ spaces, $1 < p < \infty$. The Theorems are stated here without proofs. The full version of this paper, including detailed technical proofs of the Theorems will be published elsewhere.



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## 1. INTRODUCTION

Let $E$ be a real normed linear space with dual $E*$. A mapping $A$ with domain $D(A)$ and range $R(A)$ in $E$ is called *accretive* if for all $x,y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

(1.1) $$\text{h}Ax - Ay, j(x-y)\text{i} \geq 0,$$

where $J\colon E \to 2^{E*}$ is the normalized duality map on $E$. The operator $A$ is said to be *$m-$accretive* if $A$ is accretive and the range of $(I + sA)$ is all of $E$ for some $s > 0$. It can be shown that if $R(I + sA) = E$ for *some* $s > 0$, then this holds for *all* $s > 0$. The operator $-\Delta$, where $\Delta$ denotes the Laplacian is an $m-$accretive operator.

The accretive operators were introduced independently in 1967 by Browder [1] and Kato [5]. Interest in such mappings stems mainly from their firm connection with the existence theory for nonlinear equations of evolution in Banach spaces. It is well known that many physically significant problems can be modelled in terms of an initial value problem of the form $\frac{du}{dt} + Au = 0,\ u(0) = u_0$, where $A$ is accretive in an appropriate real Banach space. Typical examples of such *evolution equations* are found in models involving the heat, wave or Schr¨odinger equation (see e.g., Browder [2]). At equilibrium, $\frac{du}{dt} = 0$, so that the evolution equation reduces to the equation:

(1.2) $$Au = 0,$$

*whose solutions correspond to the equilibrium state of the system*.

Let $H$ be a real Hilbert space and let $f\colon H \to \mathbb{R}$ be a convex functional. Consider the *subdifferential,* $\partial f\colon H \to 2^H$ defined by

$$\partial f(x) = \left\{x^* \in H : f(y) \quad f(x) \geq \langle y \quad x, x^*\rangle \ \forall\, y \in H\right\}$$

*It is known that (e.g., Rockafellar[8]) and that∂f is a maximal monotone (0 ∈∂f,∂f(then 0u) if and only*

*ifm∈−∂fmonotone) operator(u) reduces to theu is a critical point of f. Thus, if we set A:= inclusion*

*(1.3)* $0 \in Ax, x \in H,$

*whose solutions correspond to the critical points of f.*

*Since the operator A in equation (1.2) or inclusion (1.3) is generally nonlinear, considerable research* efforts have been focused on iterative methods for approximating solutions of (1.2) or (1.3), assuming existence, when the operator *A* is of the accretive (or monotone) type.

Let $N(A) := \{x \in E: A_x = 0\} \neq \emptyset$. An operator $A:D(A) \subset E \to E$ is called *generalized Φ−quasi-accretive* if any $x \in D(A)$, $x* \in N(A)$, there exist $j(x- x*) \in J(x- x*)$ such that $[Ax-Ax*, j(x- x*)] \geq \Phi(\|x - x*\|)$.

The most general results for approximation of fixed points of *uniformly continuous* Φ− *pseudo-contractive-type mappings* seem to be the following theorems.

**Theorem G1 (**[7]**, Theorem 2.1)** *Let E be a real normed linear space, K be a nonempty subset of E and T: K → E be a uniformly continuous Φ−pseudo-contractive-type operator, i.e., there exist* $x_* \in K$ *and a strictly increasing function* $\Phi : [0,\infty J) \to [0,\infty)$, $\Phi(0) = 0$h *such that for all* $x \in K$, *there exist* $j(x- x*) \in \boldsymbol{J}(x- x*)$ satisfying $[Tx - x^*, j(x- x^*)] \leq \| x- x^*\|^2 - \Phi(\|x- x^*\|)$. (*a*) if $y^* \in K$ is a fixed point in *K of T, then* $y^* = x^*$, *so T has at most one fixed point in K; (b) Suppose there exists* $x_0 \in K$ *such that both the ishikawa iterative sequence* $\{x_n\}$ *with error and the auxiliary* $y_n = (1- \beta_n) x_n + \beta_n Tx_n + v_n, n \geq 0, x_{n+1} = (1 - \alpha_n) x_n + \beta_n Tx_n + v_n, n \geq 0,$ $x_{n+1} = (1 - \alpha_n) x_n + \alpha_n Ty_n + u_n$, *are contained in K, where* $\{u_n\}, \{v_n\}$ *are two sequences in E and* $\{\alpha_n\}, \{\beta_n\}$ *are two sequences in* { 0, 1} satisfying the following conditions: (*i)* $\alpha_n, \beta_n \to 0$ $(n \to \infty)$ and $\Sigma\alpha_n = \infty$; (*ii*) $\|v_n\| \to 0(n - \infty)$. **If** $\{x_n\}$ **is a bounded sequence in** *K*, then $\{x_n\}$ converges strongly to $x^*$. In particular, *if* $y^*$ *is a fixed point of T in K, then* $\{x_n\}$ converges to $y^*$.

*Remark* 1.2. The conditions on {*un*}, {*vn*} and {*xn*} in the Theorem 2.2 of [7] are as in Theorem G1.

**Theorem C1 (**[6]**, Theorem 7.2.1, p.248)** *Let E be a real normed linear space, K be a nonempty convex subset of E such that* $K+K \subset K$, *and* $T: K \to K$ *be a uniformly continuous and* Θ*–hemi-contractive mapping. Let* $\{\alpha_n\}, \{\beta_n\}$ *be two real sequences in* (0, 1) *satisfying the following conditions: (i)* $\alpha_n, \beta_n \to 0$ $(n - \infty)$, (*ii*) $\Sigma \alpha_n = \infty$. *Assume that* $\{u_n\}, \{v_n\}$ *are two sequences in K satisfying the following conditions:* $u_n = u'_n + u''_n$ *for any sequences* $\{u'_n\}, \{u''_n\}$ *in K with* $\Sigma\|u'_n\| < \infty$; $\| = 0(\alpha_n)$ *and* $\|u_n\| \to 0$ *as* $n \to \infty$. *Define the Ishikawa iterative sequence with mixed errors in K by* $x_0 \in K$, $y_n = (1-\beta)x_n + \beta_n Tx_n + v_n, n \geq 0, x_{n=1} = (1- \alpha_n)x_n + \alpha_n Ty_n + u_n$. *If* $\{Ty_n\}$ *is bounded, then the sequence* $\{x_n\}$ *converges strongly to the unique fixed point of T.*

*Remark* 1.3. The conditions in $\{u^n\}, \{v^n\}$ in Theorem 7.2.2 of [6] are as in Theorem C1. In that theorem, the sequence $\{Ty_n\}$ is required to be bounded.

*Remark* 1.4.

1. Theorems G1 and C1 are important generalizations of several results. We observe that the lass of mappings considered in theorem C1 is a *proper subclass* of the class of mappings studied in theorem G1 in which $\Phi(s) = s\varphi(s)$. However, the requirement that $\{x_n\}$ be bounded imposed in

theorem G1 is stronger than the requirement that $\{Ty_n\}$ be bounded imposed in theorem C1. Furthermore, the boundedness of the sequences $\{x_n\}$, $\{Ty_n\}$ or $\{Sy_n\}$ imposed in these papers ([6], [7]) are difficult to verify, *ab initio*, in any possible applications.

2. It is well known that if an algorithm is applicable in establishing that the sequence of the algorithm converges strongly to a solution of a problem, the discussion of a more cumbersome algorithm for the same problem is completely undesirable unless, of course, if the cumbersome algorithm has significant advantages over the simpler one. For example, if the cumbersome algorithm converges significantly faster, or has a better error estimate. In particular, whenever the Mann algorithm works, any discussion of the so-called Ishikawa method is totally unnecessary.
3. The addition of *bounded* error terms to either the Mann or the Ishikawa-type recursion formula leads to no generalization. The proofs of theorems in this case amount virtually to unnecessary repetition of the computations employed in the case when no error terms are included (see e.g., [3], Theorem 9.1, p.130). In fact, in Theorems G1 and C1 and other theorems of [6] and [7], the conditions imposed on the error terms are *impossible* to verify in any possible application because those terms are not known precisely.
4. Existence theorems are generally proved in reflexive real Banach spaces. Consequently, it suffices to approximate solutions in such spaces.

5. It is known that uniformly continuous maps defined on normed linear spaces are bounded (see e.g., a proof in Chidume *et al.*, [4]).

Let $E$ be a real normed linear space in which the single-valued normalized duality map is Lipschitzian on balls and let $A: E \to E$ be a bounded generalized $\Phi$−quasi-accretive map.

We announce here that we have been able to construct an efficient Mann-type iterative algorithm and prove that the sequence of our algorithm converges strongly to the unique zero of $A$. Our theorems are applicable in $q$ uniformly smooth real Banach spaces, $q > 1$. These spaces include the $L_p$ spaces, $1 < p < \infty$. More precisely, the following theorems are obtained.

## 2. MAIN RESULTS

**Theorem 2.1.** *Let $E$ be a real normed linear space in which the single valued normalized duality map is Holder continuous on balls and let $A: E \to E$ be a bounded generalized $\Phi$−quasi-accretive map. For arbitrary $x_0 \in E$, define a sequence $\{x_n\}$ by*

(2.1) (2.1) $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S x_n \; n \geq 0,$

*where $S: E \to E$ is defined* $Sx = x - Ax \ \forall \ x \in E,$ *and* $\{\alpha_n\}_{n=1}^{\infty}$ *by*

*is a sequence in* $(0, 1)$ *satisfying the following conditions:* $\lim \alpha_n$ $(ii) \sum_{n=1}^{\infty} \alpha_n = \infty$ $= 0;$

*Then, there exists there exists $\gamma_0 \in \mathrm{R}^+$ such that if $\alpha_n \leq \gamma_0 \ \forall \ n \geq 0$, $\{x_n\}$ converges strongly to the unique solution of the equation $Au = 0$.*

**Theorem 2.2.** *Let $E$ be a $q$-uniformly smooth real Banach space, $q > 1$, and $A: E \to E$ be a bounded generalized $\Phi$−quasi-accretive map. For arbitrary $x_0 \in E$, define a sequence $\{x_n\}$ by*

(2.2) (2.2) $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Sx_n \; n \geq 0,$

*where S: E → E is defined* $Sx = x - Ax \; \forall \; x \in E,$ *and* $\{\alpha_n\}_{n=1}^{\infty}$ *by*

*sequence in* (0,1) *satisfying the following conditions:* $(i) \lim_{n\to\infty} \alpha_n = 0$; *(ii)* $(ii) \sum_{n=1}^{\infty} \alpha_n = \infty$. *Then there exists* $\gamma_0 \in \mathbb{R}^+$ *such that if* $\alpha_n \leq \gamma_0 \; \forall \; n \geq 0$, $\{x_n\}$ *converges strongly to the unique solution of the equation* $Au = 0$.

**Corollary 2.3.** Let $E := L_p$, $1 < p < \infty$ and $A : E \to E$ be a bounded generalized Φ−quasi-accretive map. For arbitrary $x_0 \in E$, define a sequence $\{x_n\}$ by

(2.3) $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Sx_n \; n \geq 0,$

where $S: E \to E$ is defined by $Sx = x - Ax \; \forall \; x \in E$, and $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in (0,1) satisfying the following conditions: ($i) \lim_{n\to\infty} \alpha_n = 0$; $(ii) \sum_{n=1}^{\infty} \alpha_n = \infty$. Then, there exists $\gamma_0 \in \mathbb{R}^+$ such that if $\alpha_n \leq \gamma_0 \; \forall \; n \geq 0$, $\{x_n\}$ converges strongly to the unique solution of the equation $Au = 0$.

*Remark* 2.4. Our results in this paper, under the setting of our theorems, are significant improvements of Theorems G1 and C1 and a host of other recent results in the following sense:

1. Since uniformly continuous maps defined on normed linear spaces are bounded, it follows that the class of *bounded* mappings considered in our theorems is much larger than the class of *uniformly continuous* maps considered in Theorems G1 and C1
2. The recursion formula of the Mann-type considered in our paper is much simpler and more efficient than the Ishikawa-type recursion formulas considered in these papers.
3. The boundedness of the sequence $\{x_n\}$ of the cumbersome Ishikawatype recursion formula, and for the restricted uniformly continuous maps imposed in Theorem G1 and in Theorem 2.1 of [7] is dispensed with in our theorems for the more general class of bounded maps, and superior Mann-type iteration formula. Similarly, the boundedness condition on the sequence $\{Ty_n\}$ imposed in Theorems C1 and on $\{Sy_n\}$ imposed in Theorem 7.2.2 of [6] is dispensed with in our theorems.

The full version of this paper including the technical proofs of the above theorems and corollary will be published elsewhere.

Prototype of the iteration parameter in all our theorems is $\alpha_n = \frac{\gamma_0}{1+n} \; \forall \; n \geq 0$.

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